Double squeezing in generalized q-coherent states



1994 J. Phys. A: Math. Gen. 27 L15

(http://iopscience.iop.org/0305-4470/27/2/003)

J. Phys. A: Math. Gen. 27 (1994) L15–L19. Printed in the UK

LETTER TO THE EDITOR

# Double squeezing in generalized $q$-coherent states

Roger J McDermott and Allan I Solomon
Faculty of Mathematics, Open University, Walton Hall, Milton Keynes, MK7 6AA, UK

Received 25 October 1993

**Abstract.** Using a generalization of the $q$-commutation relations, we develop a formalism in which we define generalized $q$-bosonic operators. This formalism includes both types of the usual $q$-deformed bosons as special cases. The coherent states of these operators show interesting and novel noise reduction properties including simultaneous squeezing in both field components, unlike the conventional case in which squeezing is permitted in only one component. This also contrasts with the usual quantum group deformation which also only permits one-component squeezing.

Since the advent of the theory of parastatistics [1, 2], there have been many attempts to generalize the canonical commutation relations. Motivation for such work has come from such diverse areas as resonance theory [3, 4], intermediate statistics [5], Lie-admissible models [6], non-unitary time evolution and quantum dissipative systems [7, 8]. However, the real impetus in the recent study of deformed commutation relations has been the discovery of quantum groups and algebras [9–11] and the role played by the $q$-deformed boson [12–14] in their representation theory. Since that time, there has been a lot of attention paid to possible physical applications of such $q$-oscillators (e.g. [15–18]) as well as deformations which involve more than one parameter [19] or are dependent on a deformation function [20]. Recent work has shown the applicability of such deformed oscillator techniques to many phenomenological models in such fields as atomic and nuclear physics [21], quantum optics [22] and superintegrable systems [23].

In this letter, we investigate the properties of a generalized $q$-oscillator where the degree of deformation is functionally dependent on the number operator. By considering expectation values in analogues of coherent states, we see that certain classes of deformed oscillator have non-standard quantum noise properties. Applying this technique to the two types of $q$-bosons studied in connection with quantum groups, it is seen that, whereas the *physics*-type boson has standard noise properties, the *maths*-type boson exhibits simultaneous noise reduction in both quadratures compared to the vacuum value.

We use the single-particle deformed commutation relation [24]

$$aa^\dagger - f(N)a^\dagger a = 1 \qquad (1)$$

where $a^\dagger$ and $a$ are generalized creation and annihilation operators, $N$ is the number operator such that $N|n\rangle = n|n\rangle$, $f$ is a real function, and the vacuum $|0\rangle$ is defined by $a|0\rangle = 0$. This deformation scheme includes the various forms hitherto defined in the literature.

*Examples:*

(i) $f(N) = 1$.
This is the usual commutation relation of the Heisenberg–Weyl algebra and describes ordinary quantum mechanical bosonic systems such as the the harmonic oscillator.

 

(ii) $f(N) = q$.

The so-called $q$-oscillator, first suggested by Arik and Coon [4]. It has since been studied in detail by several authors, e.g. Jannussis *et al* [7], and Kulish and Damaskinsky [14].

(iii) $f(N) = (q^{N+2} + 1)/(q(q^N + 1))$.

This gives a deformed commutation relation equivalent to that of the $q$-boson first discovered by Macfarlane [12] and Biedenharn [13] in connection with the representation theory of quantum groups.

(iv) $f(N) = (F(N+1) - 1)/F(N)$

where $F(N)$ is an analytic function. This form of deformation can be related to the extensive work of Bonatsos, Daskaloyannis and others [20, 21, 23] on their deformed oscillator formalism as well as some later work by Jannussis (e.g. [25]).

Building up normalized eigenstates of the number operator $N$ by repeated application of the generalized creation operators in (1), we obtain

$$|n\rangle = \frac{(a^\dagger)^n}{([n]!)^{1/2}}|0\rangle \tag{2}$$

where the function $[n]$ is defined recursively by

$$[n+1] = 1 + f(n)[n] \tag{3}$$

with the initial condition $[0] = 0$.

Explicitly, we see

$$[n] = 1 + f(n-1) + f(n-1)f(n-2) + f(n-1)f(n-2)f(n-3) + \cdots + f(n-1)f(n-2)\cdots f(2)f(1) \tag{4}$$

$$= \sum_{k=0}^{n-1} \frac{f(n-1)!}{f(k)!}. \tag{5}$$

The functions $[n]$ can be thought of as generalizations of the basic numbers of $q$-analysis [26]. They obey a highly nonlinear arithmetic but for appropriate choice of the function $f$ they tend in some limit to the ordinary integers.

We first consider conventional coherent states of the oscillator which obey the undeformed commutation relation ($f(N) = 1$) and may be defined by

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{6}$$

or equivalently

$$|\lambda\rangle = \exp(|\lambda|^2)^{-1/2} \exp(\lambda a^\dagger)|0\rangle \tag{7}$$

where the exponentional function, by definition, has the property

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(\lambda x) = \lambda \exp(\lambda x). \tag{8}$$

These definitions of coherent states have been used to generalize the concept to the cases where the commutation relations have been deformed.

Given the $q$-commutation relation $aa^\dagger - q\,a^\dagger a = 1$, we may define coherent states $|\lambda\rangle$ by

$$a|\lambda\rangle = \lambda|\lambda\rangle. \tag{9}$$

To achieve the alternative definition given by (7), it is necessary to introduce a $q$-derivative operator ${}_qD_x$ [26] such that

$${}_qD_x E_q(\lambda x) = \lambda E_q(\lambda x) \tag{10}$$

where $E_q(x)$ is the Jackson $q$-exponential [27]. When this is done, we see that

$$|\lambda\rangle = E_q(|\lambda|^2)^{-\frac{1}{2}} E_q(\lambda a^\dagger)|0\rangle. \tag{11}$$

The same procedure can also be used to define $q$-coherent states for the Macfarlane–Biedenharn oscillator (although in this case the generalization of the exponential function is different to that of Jackson).

For $[n]$, an analytic function of the variable $n$, defined by (5), it is possible to extend the above analysis to the case of bosonic creation and annihilation operators obeying the general commutation relations (1).

We define an operator $D_x$ such that

$$D_x = \frac{1}{x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]. \tag{12}$$

This acts as a generalized derivative operator, e.g.

$$D_x x^n = [n]x^{n-1}. \tag{13}$$

The eigenfunctions of $D_x$ given by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{14}$$

are well defined provided the function $f$ satisfies appropriate convergence criteria. If $f(n) \geqslant 1$ as $n \to \infty$ then $E(x)$ converges for all real values of $x$. If $f(n) < 1$ as $n \to \infty$, then convergence is ensured for a range of $x$ dependent on the function $f$.

Since $aE(\lambda a^\dagger)|0\rangle = \lambda E(\lambda a^\dagger)|0\rangle$, we can use $E(x)$ to define analogues of coherent states as normalized eigenstates of the generalized annihilation operator:

$$|\lambda\rangle = \{E(|\lambda|^2)\}^{-\frac{1}{2}} E(\lambda a^\dagger)|0\rangle. \tag{15}$$

We now consider conventional (undeformed) bosons.

The electromagnetic field components $x$ and $p$ are given by

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad \text{and} \qquad p = \frac{1}{\mathrm{i}\sqrt{2}}(a - a^\dagger). \tag{16}$$

As usual, we define the variances $(\Delta x)$ and $(\Delta p)$ by

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 \qquad \text{and} \qquad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2. \tag{17}$$

In the vacuum state

$$(\Delta x)_0 = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad (\Delta p)_0 = \frac{1}{\sqrt{2}} \tag{18}$$

and so

$$(\Delta x)_0(\Delta p)_0 = \tfrac{1}{2}. \tag{19}$$

The commutation relation for $a$ and $a^\dagger$ leads to the following uncertainty principle:

$$(\Delta x)(\Delta p) \geqslant \tfrac{1}{2}|\langle[x, p]\rangle| = \tfrac{1}{2}. \tag{20}$$

Thus the vacuum state attains the lower bound for the uncertainty, as do the coherent states.

While it is impossible to lower the product $(\Delta x)(\Delta p)$ below the vacuum uncertainty value, it is nevertheless possible to define *squeezed* states [29] for which (at most) one quadrature lies below the vacuum value, i.e.

$$(\Delta x) < (\Delta x)_0 = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad (\Delta p) < (\Delta p)_0 = \frac{1}{\sqrt{2}}. \tag{21}$$

If we now consider the generalized bosonic operators given by (1), using the same definitions for the the field quadratures $x$ and $p$ as in (16), we find that, just as in the conventional case, the vacuum uncertainty product $(\Delta x)_0(\Delta p)_0 = \frac{1}{2}$ is a lower bound for all *number* states. *However, unlike the conventional case, it is not a global lower bound.*

Consider the quadrature values in eigenstates of the generalized annihilation operator. We then have

$$\langle x\rangle_\lambda = \langle\lambda|\frac{1}{\sqrt{2}}(a^\dagger + a)|\lambda\rangle = \frac{1}{\sqrt{2}}(\lambda + \bar{\lambda}) \tag{22}$$

and

$$\langle x^2\rangle_\lambda = \langle\lambda|\tfrac{1}{2}((a^\dagger)^2 + a^2 + a^\dagger a + aa^\dagger)\,|\lambda\rangle \tag{23}$$

$$= \tfrac{1}{2}\{(\bar{\lambda} + \lambda)^2 + 1 - \varepsilon_{f,\lambda}|\lambda|^2\} \tag{24}$$

where

$$\varepsilon_{f,\lambda} = 1 - \langle f(N+1)\rangle_\lambda. \tag{25}$$

If we choose $0 < f(n) < 1$, then it can be shown that $\varepsilon_{f,\lambda}|\lambda|^2 \in (0, 1)$ for $\lambda$ within the radius of convergence of the generalized exponential (14).

Hence

$$(\Delta x)^2_\lambda = \tfrac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\}. \tag{26}$$

Evaluating the variance for the other component, we find that $(\Delta p)^2_\lambda = (\Delta x)^2_\lambda$, so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \tfrac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} < \tfrac{1}{2}. \tag{27}$$

However, it can also be shown that

$$\tfrac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} = \tfrac{1}{2}|\langle[x, p]\rangle_\lambda| \tag{28}$$

so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \tfrac{1}{2}|\langle[x, p]\rangle_\lambda|. \tag{29}$$

Thus we see that these generalized $q$-coherent states satisfy a restricted form of the minimum uncertainty property (MUP) of the conventional coherent states. Additionally we see that there is a general noise reduction in both quadratures compared to their vacuum value. In conventional coherent states there is no noise reduction relative to the vacuum value. In conventional squeezed states, there is noise reduction in only one component.

We can apply the preceding analysis to the $q$-deformed bosons studied recently in connection with quantum groups (see e.g. [9, 14, 15, 22]).

(i) *'Physics' q-bosons.* First consider the $q$-bosons described by Macfarlane [12] and Biedenharn [13]. These use the definition of the generalized number, $[n]$, discussed recently in the physics literature and so will be termed *'physics'* $q$-bosons. They are characterized by the deformed commutation relation

$$aa^\dagger - q\,a^\dagger a = q^{-N}. \tag{30}$$

This can be rewritten [24] as

$$aa^\dagger - f(N)\,a^\dagger a = 1 \tag{31}$$

where $f(N) = (q^{N+2} + 1)/(q(q^N + 1))$.

In this case, for normalizable eigenstates, the function $\varepsilon_{f,\lambda}$ is negative and so simultaneous two-component noise reduction does not take place. This is in agreement with the findings of Katriel and Solomon [15] and Chiu *et al* [30]. However, it can be shown that ordinary *squeezing*, i.e. noise reduction in one component compared to the vacuum (with a corresponding noise amplification in the other component), does take place [31, 32].

(ii) *'Maths' q-bosons*. We now consider the $q$-boson described by Arik and Coon [4]. This uses the generalized number function found in classical $q$-analysis and will therefore be termed a '*maths*' $q$-boson. It is characterized by the deformed commutation relation

$$aa^{\dagger} - q\, a^{\dagger}a = 1. \tag{32}$$

For $q \in (0, 1)$, the Jackson $q$-exponential $E_q(|\lambda|^2)$ converges, provided $\varepsilon_q|\lambda|^2 = (1-q)|\lambda|^2 < 1$. Given this condition on $\lambda$, we have normalizable $q$-analogue coherent states satisfying (6) in which

$$(\Delta x)^2_\lambda = (\Delta p)^2_\lambda = (\Delta x)_\lambda(\Delta p)_\lambda = \tfrac{1}{2}\{1 - \varepsilon_q|\lambda|^2\} < \tfrac{1}{2}. \tag{33}$$

Hence, for this type of $q$-boson, we do obtain noise reduction in both quadratures with respect to the vacuum value.

## References


[1] Green H S 1953 *Phys. Rev.* **90** 270
[2] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Berlin: Springer)
[3] Coon D D, Baker M and Yu S 1972 *Phys. Rev.* D **5** 1429
[4] Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524
[5] Kuryshkin V V 1980 *Ann. Fond. L. de Broglie* **5** 111
[6] Santilli R M 1978 *Hadron J.* **1** 754
[7] Jannussis A, Papaloucas L C and Siafarikas P D 1980 *Hadron J.* **3** 1622
[8] Jannussis A, Brodimas G and Mignani R 1991 *J. Phys. A: Math. Gen.* **24** L775
[9] Drinfeld V G 1987 *Proc. Int. Congr. of Math. (Berkeley, 1986)* pp 798–820 (AMS)
[10] Manin Yu I 1988 *Montreal Univ. preprint* CRM-1561
[11] Woronowicz S L 1987 *Comm. Math. Phys.* **111** 613
[12] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[13] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[14] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** 415
[15] Katriel J and Solomon A I 1991 *J. Phys. A: Math. Gen.* **24** 2093
[16] Birman J L 1992 *Phys. Lett.* **167A** 363
[17] Bogoliubov N M and Bullough R K 1992 *Phys Lett.* **168A** 264
[18] Shabanov S V 1993 *J. Phys. A: Math. Gen.* **26** 2583
[19] Chakrabarti R and Jagannathan R 1991 J. Phys. A: Math. Gen. **24** L711
[20] Daskaloyannis C 1992 *J. Phys. A: Math. Gen.* **25** 2261
[21] Bonatsos D and Daskaloyannis C 1992 *Phys. Lett.* **278B** 1
[22] Chaichian M, Ellinas D and Kulish P P 1990 *Phys. Rev. Lett.* **65** 980
[23] Bonatsos D, Daskaloyannis C and Kokkotas K 1994 1993 *Proc. 3rd International Wigner Symposium (Oxford, 1993)* at press
[24] Solomon A I and Birman J L 1993 *Symmetries in Science VII* ed B Gruber (New York: Plenum)
[25] Jannussis A 1993 *J. Phys. A: Math. Gen.* **26** L233
[26] Jackson F H 1909 *Mess. Math.* **38** 62
[27] Exton H 1983 *q-Hypergeometric Functions and Applications* (Chichester: Ellis Horwood)
[28] Glauber R J 1963 *Phys. Rev.* **131** 27766
[29] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[30] Chiu S H, Gray R and Nelson C A 1992 *Phys. Lett.* **164A** 237
[31] Buzek V 1991 *J. Mod. Opt.* **38** 801
[32] Celeghini E, Rasetti M and Vitiello G 1991 *Phys. Rev. Lett.* **66** 2056